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The CES production function with its estimation techniques

by

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The pioneer work of Arrow, Chenery, Minlas and Solow (1961) stimulated many economists to work in the field of production functions both empirically and theoretically. The writer believes that the CES (constant elasticity of substitution) production function provides an ideal point of departure for Thai students who wish to gain an insight into this area of work. The study of the CES production function world facilitate understanding of later developments in the form of more general production relationships such as the VES (variable elasticity of substitution) and translogarithmic production functions.

Production Function

It is difficult to deny Sammelson's assertion:

'Until the laws of thermodynamics are repealed I shall continue to relate outputs to inputs,' (11) That is, the quantity of output is necessarily constrained by the available supplies of capital and labor, a relationship which can simply be written as:-

$$V = F(K,L)$$

where V is the output, K and L are the amounts at capital and labor respectively. This functional relationship embodies certain inherent difficulties. Firstly, the use of an aggregate production function is the subject of much controversy. Further, K and L are sometimes interpreted as stocks and sometimes as **flows** of capital and labor services. (5)

The Elasticity of Substitution

The concept of substitution has been utilized by the neo-classical economists as a basis of both production and distribution-theories. Although an interest in the elasticity of substitution existed even prior to the introduction of the CES production function by Arrow, Chenery, Minlas and Solow (ACMS), it was only after the CES production function was introduced that a wave of literature involving the elasticity of substitution occurred. Prior to that the two most common production functions are the Cobb-Douglas and the Fixed-coefficients ('Leontief') production functions. The former assumed a priori unitary elasticity of substitution while the latter assumed a priori zero elasticity of substitution. The CES production function is, therefore, a general function representing a family of production functions in which the elasticity of substitution is an unspecified constant. Moreover, the Cobb-Douglas and the fixed coefficient production functions appeared as special cases of the CES function (1).

The elasticity of substitution concept was first developed by J.R. Hicks (6). It measures the degree to which the substitutability of one factor for another varies as the proportion between the factors varies.

The elasticity of substitution also shows how rapidly diminishing returns to one factor input set in when its price falls relative to the other factor price.

If the function is

$$V = F(K,L)$$

then the elasticity of substitution can be written symbolically as:

$$\sigma = \frac{L/K \; d(K/L)}{\frac{1}{MRS} \; dMRS}$$

Thus the elasticity of substitution is the proportional change in the relative factor input to a proportional change in the marginal rate of substitution (MRS) between labor and capital (or proportional change in the relative factor price ratio).

The MRS of labor for capital is the ratio of the marginal product of labor to the marginal product of capital. We can write it as follows:

$$MRS = \frac{\partial Q}{\partial L} \Big/ \frac{\partial Q}{\partial K}$$

The elasticity of substitution is independent of the units of measurement of the labor and capital inputs. (6).

There are at least four important areas in which the elasticity of substitution plays a significant role. These areas are as follows:

1. The stability or instability of certain growth paths implied by some models, for example the Harrod-Domar model, depends on the value of the elasticity of substitution.

2. The effects of varying factor endowments on the pattern of trade and relative factor prices depends heavily on the nature of variation in the elasticity of substitution between factors among different industries.

3. ACMS restate the traditional importance of the elasticity of substitution for relative shares over time

4. Knowledge of the values of the elasticity of substitution in the industrial and agricultural sectors (indual economies) can be useful for policy makers in manipulating? market signals to ensure greater labor absorption.

Thus the concept of the elasticity of substitution is of considerable importance in questions of policy determination. The implications are clear when we consider the impact of factor price variations. As the elasticity of substitution approaches 0 it is increasingly difficult to substitute one factor for another. Consequently, labor cost variations(but for the case of a homogeneous labor force) will not necessarily attract new firms to the locality with the lowest wage rate. Here the calculation will necessitate an overall view of the cost of the remaining factors of production. This is particularly true in the area with high unemployment (rural on urban) (where wage variations occur? Capital appears to be more scarce and consequently more expensive due to the greater risk involved.

Then, variation among industries of the elasticity of substitution can lead to substantial changes in relative factor proportions as the price of labor rises in comparison to the price of capital. These changes in turn affect relative prices, regional trade, and the sectoral distribution of employment. All of these are important in planning for future growth.

Derivation of the CES production function

From the neoclassical assumption of linearly homogeneous production functions, we derive the following form:

$$y = f(x)$$

where $y = \frac{V}{L}$, $x = K/L$, $V = F(K,L)$ Since

$$\sigma = -\frac{f'(x)\,[f(x)-x\,f'(x)]}{x\,f(x)\,f''(x)} \qquad (1)$$

$$= \frac{dy}{d\omega} \cdot \frac{\omega}{y}$$

$$\text{or } y = a\,\omega^{\sigma} \qquad (a = \text{a constant})$$

By using the logarithmic transformation, we obtain

$$\log y = \log a + b \log(y - x\frac{dy}{dx}), \quad (f'(x) = \frac{dy}{dx})$$

$$\text{or } y = a\left(y - x\frac{dy}{dx}\right)^b$$

$$y^{1/b} = a^{1/b}\left(y - x\frac{dy}{dx}\right)$$

$$\left(\frac{y}{a}\right)^{1/b} = y - x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{a^{1/b}y - y^{1/b}}{a^{1/b}x} = \frac{y(1-\alpha y^{\rho})}{x}$$

where $\alpha = a^{-1/b}$, $\rho = \frac{1}{b} - 1$. Then

$$\frac{dx}{x} = \frac{dy}{y(1-\alpha y^{\rho})} = \frac{dy}{y} + \frac{\alpha y^{\rho-1}\,dy}{1-\alpha y^{\rho}}$$

$$= \frac{dy}{y} + \frac{1}{\rho}\left(\frac{\alpha\rho y^{\rho-1}\,dy}{1-\alpha y^{\rho}}\right)$$

Transforming the above equation to the logarithmic form by integration we obtain:

$$\log x = \log y - \frac{1}{\rho}\log(1-\alpha y^{\rho}) + \frac{1}{\rho}\log\beta$$

when β is any constant. The functional form of this logarithmic function is:

$$x^{\rho} = \frac{y^{\rho}\beta}{1-\alpha y^{\rho}}$$

$$y = x[\beta + \alpha x^{\rho}]^{-\frac{1}{\rho}}$$

$$= (\beta x^{-\rho} + \alpha)^{-\frac{1}{\rho}}$$

$$v = L(\beta K^{-\rho} L^{\rho} + \alpha)^{-\frac{1}{\rho}}$$

$$\text{or} \quad v = (\beta K^{-\rho} + \alpha L^{-\rho})^{-\frac{1}{\rho}}$$

This last equation is the famous CES production function.

The CES production function

$$v = A[\delta K^{-\rho} + (1-\delta) L^{-\rho}]^{-\frac{\mu}{\rho}} \qquad (1)$$

where A is an efficiency parameter which changes output for given quantities of input? δ ($0 \leq \delta \leq 1$) is a distribution parameter which determines the division of factor income. ρ is a substitution parameter and it is related to the elasticity of substitution parameter as follows:

$$\sigma = \frac{1}{1+\rho}, \quad 1 \leq \rho \leq \alpha$$

since $0 \leq \sigma \leq \infty$ and μ is the degree of homogeneity parameter, such that when $\mu > 1$ there exists increasing returns to scale. If we take the logarithmic transformation of (1) we obtain.

$$\log V = \log A - \frac{\mu}{\rho} \log[\delta K^{-\rho} + (1-\delta) L^{-\rho}] \qquad (2)$$

Equation (2) is not linear. Thus, in order to estimate the parameters it is necessary to use a nonlinear least squares method. An alternative method of estimation, based on linear least-squares estimation was recently suggested by Kmenta (10). Thus one can expend

$$\log (\delta K^{-\rho} + (1-\delta) L^{-\rho})$$

by Taylor's series around $\rho = 0$ and disregard terms of third and higher orders, so that equation (2) becomes:

$$\log V = \log A + \mu\,\delta \log K + \mu\,(1-\delta)\log L - \frac{1}{2}\rho\mu\delta\,(1-\delta)\,[\log K - \log L]^2 + e \quad (3)$$

(Perhaps, it should be mentioned that the first 3 terms are in fact the Cobb-Douglas f." and the term $-\frac{1}{2}\rho\mu\delta$ ...... is a correction for $\rho \neq 0$) from which, in contrast to (2), it is possible to compute estimates of A, μ, δ and ρ by linear least-squares regression.

Estimation techniques

Method I (ACMS's technique)

The elasticity of substitution parameter can be estimated quite easily if we assume competitive markets and constant returns to scale, i.e., $\mu = 1$.

Under these assumptions Equation (1) can be reduced to

$$\log \frac{V}{L} = A + b \log \omega + \log e \quad (4)$$

where the coefficient $b = \frac{1}{1+\rho} = \sigma$ and the intercept

$A = \frac{1}{1+\rho}(\log [A^{\rho}(1-\delta)^{-1}])$. Equation (4) is linear.

Thus by simply regressing the logarithmic transformation of output per employee, $\log \frac{V}{L}$, on the logarithmic transformation of the real wage per unit flabor, $\log\omega$, we can obtain directly the elasticity of substitution parameter as the coefficient b.

A test of the two fundamental assumptions underlying this model - constant returns to scale and perfect competition can be conducted by simply testing an alternative relationship:

$$\log \frac{V}{K} = A_1 + b_1 \log r + \log e \qquad (5)$$

where $\log \frac{V}{K}$ is the logarithmic transformation of output per unit of capital and log r is the logarithmic transformation of the rental of capital. The coefficient $b_1 = \frac{1}{1+\rho} = \sigma$ and the intercept $A_2 = \frac{1}{1+\rho}$ $(\log [A^{\rho\delta}]^{-1})$. Equation (5) is linear; thus we can obtain the elasticity of substitution as the coefficient b_1 by ordinary least-squares regression.

If the perfect competition and constant returns to scale assumptions are correct then $b = b_1$. (2)

It is interesting to compare the value of the elasticity of substitution under the assumption of constant returns to scale estimated from Equations (4) and (5) with the alternative method in which the scale parameter is free to vary.

Model II (A Two - step technique)

It is possible to estimate the parameters of the CES production function by a two-step estimating technique. Moreover, the two-step procedure avoids the non-linear estimation problem of Equation (2).

From equation (1) the marginal productivities of capital and labor are respectively:

$$\frac{\partial V}{\partial K} = A\mu\delta^{-(\rho+1)}[\delta K^{-\rho} + (1-\delta) L^{-\rho}]^{-(\frac{\mu}{\rho}+1)} \qquad (6)$$

and $$\frac{\partial V}{\partial L} = A\mu\,(1-\delta)\,L^{-(\rho+1)}\,[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-(\frac{\mu}{\rho}+1)} \qquad (7)$$

If we assume competitive markets and profit-maximizing behaviours, then, for each input the value of the marginal product should be equal to the price of the input. Thus, Equation (6) and (7) are equal also to the price of capital, r, and the wage rate, ω, respectively.

Therefore, the reduced form of the ratio of Equations (6) and (7) can be rewritten as:

$$\frac{r}{\omega} = \frac{\delta}{1-\delta}\left(\frac{K}{L}\right)^{-(\rho+1)} e$$

which is linear in its logarithmic transformation

$$\log\frac{r}{\omega} = \log\left(\frac{\delta}{1-\delta}\right) - (\rho+1)\log\frac{K}{L} + \log e \qquad (8)$$

Thus we can estimate the parameter δ from the intercept, $\log\left(\frac{\delta}{1-\delta}\right)$, and ρ from the coefficient $(\rho+1)$ by the least-squares method. The elasticity of substitution parameter, σ, can now be calculated easily since ρ, as pointed out above, relates to the elasticity of substitution parameter as follows: $\sigma = \frac{1}{1+\rho}$.

Now it is simply a matter of using Equation (2) since the parameters ρ and δ mere already estimated from Equation (8) as $\hat{\delta}$ and $\hat{\rho}$

$$\log V = \log A - \frac{\mu}{\rho}\log[\hat{\delta}K^{-\hat{\rho}} + (1-\delta)L^{-\hat{\rho}}] + \log e \qquad (9)$$

The term in the brackets can be calculated first and a subsequent least-squares regression will give us an estimate of the homogeneity parameter, μ , from the coefficient $\frac{\mu}{\rho}$, so that $\hat{\mu} = \frac{\mu}{\rho}\hat{\rho}$.

Thus, in utilizing the two-step technique, the CES production function becomes linear and the two parameters--returns to scale, μ, and the elasticity of substitution, σ , can be estimated.

Model III (Dhrymes' technique)

In addition to the above techniques we will utilize another approach. Prof. Dhrymes suggests some extensions and tests for the CES class of production functions (3). He tests the assumptions of perfect competition in both the product and factor markets and the assumption of homogeneity of degree one, which are generally assumed in the development of the CES production function in the original work by ACMS (2).

Dhrymes has shown that a variant of the CES function can be written as follows:

$$\omega = A V^{\rho} L^{\gamma} \tag{10}$$

under the constraint that.

$$\gamma = \mu\rho - 1$$

so that we can obtain the following estimators for ρ^* ($\neq \rho$) and μ, i.e.,

$$\hat{\rho} = 1 - \hat{\beta}, \quad \sigma = \frac{1}{1-\rho} = \frac{1}{\beta} ; \hat{\mu} = \frac{1+\hat{\gamma}}{1-\beta}$$

where $\hat{\beta}$ and $\hat{\gamma}$ are the least-squares estimators of β and μ, respectively, obtain from the logarithmic transformation of Equation (10).

$$\log \omega = \log A + \beta \log V + \gamma \log L \qquad (11)$$

Here μ is an estimator of the homogeneity parameter and $-\frac{1}{\gamma} = \hat{\sigma}$ is the elasticity of substitution parameter. However, Dhrymes suggests that it seems preferable to derive an estimator for the elasticity of substitution parameter by using the regression.

$$\log L = -\frac{1}{\gamma} \log A + \frac{1}{\gamma} \log \omega - \frac{\beta}{\gamma} \log V \qquad (12)$$

Since "L, ω, and V are jointly dependent variables the regression will simply yield the conditional expectation of L given as and V". (3). The coefficient of the first independent variable w will thus directly give us the elasticity of substitution parameter.

Dhrymes accepts that his technique is "not a very rigorous statistical technique", but there are at least three reasons why this technique is appropriate. First, it is an additional test which can be used to compare with other techniques. Secondly, Dhrymes' technique does not require capital data which is known to be quite inadequate in most countries and finally, the additional computation required by this technique is minimal compared with the possible gain which might be derived from it.

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* Further readings in this subject can be found in "The Theory and Empirical Analysis of Production", in Studies in income and wealth., Vol.31, NBER, 1967.