Estimating the Term Structure of Interest Rates for Thai Government Bonds: A B-Spline Approach

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Abstract

B-Spline approximation technique has been applied as an empirical methodology for estimating the term structure of interest rates. This study investigates the implementation of B-spline curve fitting to the term structure of interest rates for Thai government bonds. Four fitting models namely non-restricted discount fitting, restricted discount fitting, spot fitting and forward fitting are estimated by regression equations. The generalized cross validation and mean integrated squared error are two indices applied to the selection of optimal conditions. The estimation results are then compared with the Thai Bond Market Association (ThaiBMA) zero-coupon yield curve. This study finds that the generalized cross validation and mean integrated squared error criteria provide similar term structure estimates. The ThaiBMA yield curve is also within appropriate confidence interval of estimates derived from both criteria. B-spline approximation is a robust method and good alternative of the term structure of interest rates estimation for Thai government bonds.

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Chapter 1

Introduction

Statement of Problem

Interest rates and their dynamics are a crucial part of modern financial theory. Financing and investment decisions in corporate finance are greatly dependent on interest rates. In financial engineering and portfolio management, they are used in pricing securities such as fixed-income securities, derivative securities sensitive to interest rates, and other capital assets. Other important implications are hedging long-term interest risk and developing interest rate derivative securities. Study of interest rate modeling will help better understanding of the term structure or yield curve behavior.

The term structure of interest rates represents the relationship between fixed interest securities that differ only in their time to maturity. It is simply the relationship between yields and terms. When interest rates of bonds are plotted against their terms, this is called the yield curve. In the world of finance, knowledge of the term structure is very helpful in pricing fixed coupon bonds. Knowing the discount factor curve, pricing securities at any date is simple when cashflows are available. In addition, implied forward rate curve can be derived. Mathematically, the yield curve can be used to predict interest rates at future dates. It helps to understand the future direction of interest rate movement.

In Thailand, the Thai Bond Market Association (ThaiBMA), formerly the Thai Bond Dealing Center (ThaiTBDC), provides data on the zero coupon yield curve on a daily basis. ThaiBMA interpolates the curve from average bidding yields for all bond maturities by the method of generalized bootstrapping. To decompose all coupon payments into a set of zero coupon bonds, ThaiBMA uses cubic splines and a system of simultaneous equations to derive the yield curve. For each trading day, there is a corresponding zero-coupon yield curve interpolated with a considerable accuracy. These data provide a static benchmark for the bond market. The shortfall of this method is a large number of the number of equations and variables that has to be estimated. For approximately 20 government bonds outstanding, there are more than

200 variables and equations. However, in search for more flexibility, there is a need to resort to other term structure estimation techniques such as B-spline.

Basic interest rates normally used in the term structure context are spot rate and forward rate. Discount function which is a reciprocal function of spot rate can also be easily derived. Bond pricing models used to estimate the term structure of interest rates in this study comprise of discount fitting model, spot fitting model and forward fitting model. The three models utilize their respective determinant, discount function, spot function and forward function.

Research Question

Which of the following fitting models: discount fitting model, spot fitting model, and forward fitting model, can best explain the yield curve behavior and characterize the term structure of interest rates?

Objective

To examine the extent to which the fitting models capture the movement in yield curves.

Scope of Study

Since the term structure requires short-term, medium-term and long-term government default-free fixed income instruments, yield treasury bills and government bonds with maturity ranging from 1 year to 15 years are required. Bond trading data on Friday, January 13, 2006 are obtained from the Thai Bond Market Association (ThaiBMA).

Limitation

Estimation of term structure requires zero-coupon bonds. With no existence of zero-coupon bonds in the Thai market, the term structure cannot directly be observable. Estimation results are subject to pricing error resulting from coupon effects. In addition, this study attempts to use coupon bonds as substitutes for discount bonds. Since there are many shot-term bonds but a few long-maturity bonds of 15 or more years to maturity, the availability of data set at the long-term horizon might be an obstacle for accurate fit of high-maturities term structures.

Chapter 2

Literature Review

Estimation of the term structure of interest rates requires a range of pure discount or zero-coupon bonds of different maturities. In practice, these fixed income securities are not always available. In the case of default-free bonds, for example, Thai government bonds of maturities more than one year are coupon bonds. In addition, many times these securities readily available in the bond markets do not cover the full maturity spectrum. With these limitations, mathematical and statistical techniques are needed to fit the term structure.

One of the fundamental and well-known methods for extracting yields of discount bonds from coupon bonds is bootstrapping approach. Starting with zero-coupon rates with maturities of one year or less, combined with price, coupon payment and maturity of longer maturity coupon bonds, the longer maturity yield can be derived from solving discounted cash flow equality. However, the bootstrapping method has a significant drawback. According to Choudhry (2004b) there are many reasons contributed to its inability to accurately fit the yield curve. First, in practice, it is very difficult to find a range of coupon bonds distributing coupon payments at the same time in the future. In addition, the bootstrapping technique does not allow for non-standard period calculation. Third, especially for corporate bonds, the market price of bonds incorporates other factor determinations such as liquidity, tax effect of the bond cash flows, and bid-offer spread.

A more complicated approach relies on simple linear regression. Charleton and Cooper (1976) estimated the term structure using simple linear regression on the principle that the price of bond is a function of discount factors. He utilized the fact that the U.S. treasury notes and bonds with maturities up to seven years made semi-annual payment and principal repayment on only four days in a calendar year. These four days are February 15, May 15, August 15, and November 15. With this special characteristic of even-spacing and discrete cashflow streams, discount factor coefficients can be easily estimated from the regression. However, it is apparent that this approach relies on regularly-spaced payment dates. One of the disadvantages of the Charleton and Cooper's method is that it cannot be applied to other markets such as the U.K. or Thailand where interest payment dates depend on the issuance date. Moreover, the yield curve derived from this method does not exhibit smoothness. The implied forward yield curve is rather kinked so it does not fit the term structure well.

A better accepted approach was proposed by McCulloch (1971) using polynomial splines to estimate discount function. A spline is a function that is used to approximate another function by joining points at the end of each interval. McCulloch applied quadratic splines to estimate the term structure from discount functions by imposing a major assumption that coupon payments are made continuously through time. This assumption eliminates the calculation of accrued interest under discrete-time coupon payments. The problem encountered by McCulloch was that the corresponding forward curve from the estimated discount functions exhibits what he called "knuckles". Derivation of the implied forward curve is through the first derivative of discount functions. By means of quadratic splines and the fact that forward curve can be identified as the first derivative of discount function, "knuckles" in forward curve provides discontinuous first derivative of the forward curve or discontinuous second derivative of the discount-fitted curve. This implied that the forward curve is not smooth enough to tackle this problem. McCulloch then replaced quadratic splines with cubic splines (McCulloch, 1975). Cubic splines give better flexibility to the shape of curves. However, there was no imposition of non-increasing constraint on discount function so forward rates could be negative.

Another type of polynomial used in estimating the term structure is the Bernstein polynomial. In a similar manner as McCulloch's method (1971, 1975), Schaefer (1981) estimated the discount function using Bernstein polynomials as approximation functions. The advantage is that approximations to derivative can be achieved over conventional polynomial functions. Negative forward rates can be avoided. One problem noted, nevertheless, was that Bernstein polynomial approximation functions can make the curve rise or fall sharply at long maturities.

With the problem with Bernstein polynomial acknowledged, Vasicek and Fong (1982) formed discount functions using exponential functions to produce asymptotically flat forward curves. They argued that splines have different curvature from exponentials and splines alone do not fit the discount functions well. They used cubic splines to estimate the exponentially-transformed discount function. The method was claimed to be successful, but no evidence was provided. Moreover, Shea (1985) refuted the claim that exponential splines produced more stable estimates than polynomial splines. According the Shea, piecewise polynomial splines have as good ability in fitting the term structure as exponential splines do. Exponential characteristic was again used by Chambers, Carleton and Waldman (1984) under a different approach. They applied polynomial approximation functions directly to spot curves. The spot curves are then linked to bond prices by the exponentiation of bond price function. Actual bond price data are then applied to model estimation.

Another effective choice of estimation method was mainly from the work of Shea (1984). Shea (1984) reported that some spline bases, such as those chosen by McCulloch (1971, 1975) produce a regressor matrix whose columns are nearly perfectly collinear producing inaccurate estimation. Hence, Shea recommended basis spline or B-spline method to estimate approximation function. The yield curves can be divided into non-overlapping intervals. Each interval can be estimated using approximating functions in the space that is spanned by a linear combination of basis functions. Two piecewise functions are joined at a point called knot point or knot. Specifying the appropriate number of knot points and deciding on where they should lie is no easy task. Deacon and Derry (1994) pointed out that if the number of knots is too low, the model might not provide accurate approximation under the term structure of unusual shapes. On the other hand, if the number of knots is too high, the estimated curves may be too sensitive to insignificant data points. There is no universal standard for the optimal number of knots which makes the B-spine method subjective. The rule of thumb adopted by McCulloch (1975) is to set the number of knots equal to the square root of the number of bond for the selected data sample. The width of intervals of estimation can be spaced evenly. An advantage of B-spline method is that it is easy to impose a constraint, such non-increasing discount function, on the spline function that would prevent the forward rates from being negative.

Many studies dedicated to B-spline approximation have been successfully implemented with different term structures. Steeley (1991) estimated the discount functions of gilt-edged term structure of the U.K government fixed-income securities using spline functions. His study resulted in smooth spot-rate and forward-rate yield curves. Lin and Paxon (1995) and Lin (1999) employed similar methodology to the term structure of the German and Taiwanese government bonds respectively. They also confirmed the B-spline functions as good approximations to smooth curve fitting. Moreover, Deacon and Derry (1994) cited B-splines as a preferred method for estimating the term structure of interest rates.

Another approach for curve fitting was proposed by Nelson and Siegel (1985) whose method was used and modified by many researchers. Nelson and Siegel curve is a family of curves that models implied forward-rate yield curve as a function of maturity. Their idea was based on parsimonious modeling in which few parameters can capture the whole yield curve. The implied forward rate curves generated can be of a variety of shapes. According to Jordan and Mansi (2000), the curve approaches a certain asymptotic value at the long maturity. A major drawback due to limited parameters is the flexibility trade-off (James & Webber, 2000). The yield curves can be modeled with less accuracy. In fact, in modeling discount curve, accurate results can not be obtained from this method (Choudhry, 2004a). In addition, inaccurate fit is observed at the very short end and the long end of the curve.

Other variations of Nelson and Siegel curves extended by subsequent researchers introduced some sophistication by increasing the number of parameters to be estimated. Svensson (1994) extended the work of Nelson and Siegel (1985) by adding an exponential decay term resulting in two extra parameters to capture the humped characteristic of the yield curve. Wiseman (1994) described a curve family with $n+1$ exponential decay basis elements. Bjork and Christensen (1997) substituted added coefficients in Wiseman model with polynomial functions thereby significantly raise the number of parameters to be estimated.

Nelson and Siegel curve and its variations do not require knot point specification. However, the major disadvantage is the fact that these curves are less flexible and do not capture the term structure as accurately as the spline models do.

James and Webber (2000) state that this inflexibility is due to limited parameters, 4 parameters for Nelson and Siegel curves and 6 parameters for Svensson curves. Choudhry (2004c) concludes that Nelson and Siegel and their extensions are reasonable for approximation as long as the yield curve is not too complicated. The curves derived from these methods are not appropriate for no-arbitrage applications.

This research study focuses on the application of B-spline approximation on the term structure of interest rate estimation. In particular, the foundation laid out by Lin (1999, 2002) who extracted the term structure of interest rates from Taiwanese government bonds, provides an alternative and a robust method in this area. The term structures of interest rates using different fitting model were fitted with B-spline approximation technique. The spot fitting approach provided the best fit to yield curve according to Lin (1999, 2002). Even though the Taiwanese government bond market is relatively small in comparison with other developed markets in North America, Europe and Japan, the validity of the methodology could provide a justification for application to the Thai government bond market. Furthermore, the term structure estimation using B-spline can also be extended to apply to other interest rate models such as the Heath, Jarrow and Morton (HJM) model (Health, Jarrow & Morton, 1992). Finally, it is possible to test the consistency of the curve-fitting technique to affine interest rate models proposed by Vesicek (1977) or Cox-Ingersoll-Ross (CIR) model (Cox, Ingersoll & Ross, 1985) as well.

Chapter 3

Theoretical Framework

Bond Price and Yield

A zero-coupon bond, or a zero, is the most rudimentary fixed-income securities. It is a debt instrument whose holder is entitled to receive a single payment in the future, usually of a specified amount greater than a purchase price and at a specified time. Since a zero-coupon bond does not pay interest, it is purchased at discount allowing the holder to redeem them at the higher price. For this reason, a zero-coupon bond is sometimes referred to as a discount bond.

The price of a zero-coupon bond is determined by the bond interest rate or yield. Let *P(t,T)* be the price at time t of a zero-coupon bond maturing at time T, and r be the corresponding yield or interest rate. The relationship between the price and yield of a pure discount bond at time t is

$$P(t,T) = (1+r)^{-(T-t)} \tag{1}$$

In a discrete compounding environment where interest is compounded more than once over the life of the zero-coupon bond, if the market interest rate is constant,

$$P(t,T) = \left(1+\frac{r}{m}\right)^{-m(T-t)} \tag{2}$$

where m is the number of compounding periods in one year. Although traditional fixed-income analysis assumes that interest rate compounding occurs at discrete points or over finite intervals, as compounding periods grow shorter and frequency is increasing, discrete compounding will become continuous compounding. With continuous compounding when m in Equation (2) approaches infinity and r remains constant, Equation (2) becomes

$$P(t,T) = e^{-r(T-t)} \tag{3}$$

Spot Rate

For modeling purposes, it is convenient to assume that bond prices follow a continuous-time stochastic process. Interest rates can vary through time. The rate payable at time *t* for a repayment at time $t + \Delta t$ where Δt is an incremental change in time is defined as $r(t, t + \Delta t)$. The corresponding bond price is then be

$$P(t, t + \Delta t) = e^{-r(t, t+\Delta t)\Delta t} \tag{4}$$

Rearranging Equation (4), interest rate $r(t, t + \Delta t)$ can be obtained.

$$r(t, t + \Delta t) = -\frac{\ln P(t, t + \Delta t)}{\Delta t} \tag{5}$$

As Δt becomes infinitesimally small and approaches zero, $r(t, t + \Delta t)$ becomes what is called the short rate illustrated by

$$r_t = \lim_{\Delta t \to 0} r(t, t + \Delta t) = \lim_{\Delta t \to 0} -\frac{\ln P(t, t + \Delta t)}{\Delta t} = -\frac{\partial}{\partial T} \ln P(t, t) \tag{6}$$

Short rate or short interest rate is a current and very short-term interest rate imposed on an amount of money that needs to be paid back almost instantaneously.

When incremental change is extended from Δt to $n\Delta t$, where $n = 1, 2, 3, \dots$,

$$r(t, t + n\Delta t) = -\frac{\ln P(t, t + n\Delta t)}{n\Delta t} \tag{7}$$

Let $T = t + n\Delta t$, the spot rate can be derived from Equation (7)

$$r(t, T) = -\frac{\ln P(t, T)}{T - t} \tag{8}$$

Forward Rate

Spot rate and forward rate are closely related. It is possible to extract implied forward rates from a spot yield curve. Under continuous-time environment, the forward rate $f(t,s)$ is the interest rate, which is agreed upon at time *t*, charged on funds borrowed at time *s* where $s > t$. The funds have to be paid back almost instantly and $s-t$ is infinitesimally small.

In an absence of arbitrage and uncertainty, funds invested over time $s-t$ compounded with the forward rate $f(t,s)$ must provide the same return as those invested at the spot rate $r(t,T)$.

$$e^{\int_t^T f(t,s)ds} = e^{r(t,T)(T-t)} \tag{9}$$

The expression for the spot rate in terms of the forward rate is given by

$$r(t,T) = \frac{\int_t^T f(t,s)ds}{T-t} \tag{10}$$

While in a discrete time process, one plus the spot rate is the geometric average of one plus the forward rates, The relationship in Equation (10) principally describes the spot rate *r(t,T)* as the arithmetic average of the forward rates *f(t,s)*. Rearranging the term in Equation (10) yields

$$r(t,T)(T-t) = \int_t^T f(t,s)ds \tag{11}$$

By Differentiation of Equation (11) with respect to *T*, the forward rate can be directly written in terms of the spot rate as follow.

$$f(t,T) = r(t,T) + (T-t)\frac{\partial r(t,T)}{\partial T} \tag{12}$$

It can further be noted that when $T \rightarrow t$, *r(t,T)* becomes r_t, the instantaneous spot rate or short rate in Equation (6). Combining with Equation (7), the derived relationship is then

$$r_t = -\frac{\partial}{\partial T}\ln P(t,t) = \lim_{T \rightarrow t} f(t,T) = f_t \tag{13}$$

Therefore, in a no-arbitrage environment, the instantaneous forward rate is equivalent to the instantaneous spot rate.

Bond Price as a function of spot rates and forward rates

Bond pricing concept is similar to that of equity in that the price of bond is the sum of all the discounted future cashflows. Provided that spot rates or forward rates are available, these rates can be applied to calculate bond price. For a zero-coupon bond, pricing is much less complicated due to the fact that a single payment is made at the maturity. The price at time *t* of the zero-coupon bond paying the principle of 1 at maturity time *T* can be expressed in terms of spot rates and forward rates as follow.

From the relationship in Equation (8), bond price can be expressed in terms of spot rates as

$$P(t,T) = e^{-r(t,T)(T-t)} \tag{14}$$

This is can be viewed as the generalization of Equation (3) when *r* is allowed to vary through time. Moreover, strictly for the pure discount bond with the principle of 1, Equation (14) is known as a discount factor. A collection of discount factors for all maturity is called the discount function. Attainment of discount function makes it feasible for the price calculation of coupon bonds.

In a similar manner, from Equations (3) and (10), it is possible express bond price in terms of the forward rates as

$$P(t,T) = e^{-\int_t^T f(t,s)ds} \tag{15}$$

The Term Structure Estimation

Estimation of the zero-coupon yield curves can be performed with both parametric and non-parametric approach. Parametric curves rely on term structure model such as the Vasicek model (Vasicek, 1977) or the Longstaff and Schwartz model. Non-parametric curves, on the other hand, belong to general curve-fitting approaches that does not derived from an interest rate model. They are used to calibrate interest rate models such as extended Vesicek or Health-Jarrow-Morton (HJM) interest rate model.

Other than the parametric and non-parametric classification of curves, families of curves can also be categorized as linear and non-linear. A linear family forms a vector space with a set of basis elements so that every curve in a family can be represented as a linear combination of basis functions. Because of this property, linear curves are easy to optimize to get a best fit. While linear family of curves involves a set of basis functions, a non-linear family does not. Hence, a non-linear curve cannot be represented by a combination of other basic curves. This makes the optimization troublesome and it may be difficult to get a good fit.

Two popular techniques for fitting non-parametric curves are basis splines (B-splines) and Nelson and Siegel (1985). The B-spline curves are considered to be in the linear family of curves. Nelson and Siegel curves belong to the non-linear family which provides inferior curve fitting quality. Due to the fact that this research places an emphasis on model accuracy and flexibility, B-spline is the method of choice.

A spline is a type of piecewise polynomial functions that is widely used in data interpolation and smoothing. There are several ways of applying them. The most straight forward method is regression splines, spline function fitted using regression techniques. However, regression splines may cause curves to oscillate widely and are extremely sensitive to changes in modeling parameters.

An improved technique is the smoothing spline method. It requires a parameter controlling the degree of curvature that is tolerated in a fitted curve. For the term structure estimation, this parameter should be a function of maturity.

An n^{th}-order polynomial spline introduced by McCulloch (1971, 1975) is a piecewise polynomial approximation with n-degree polynomials that are differentiable n-1 times. A yield curve can be estimated using many different polynomial splines connected at arbitrary selected points called knot points. The choice of order n widely used in the term structure and bond pricing research is the order of 3. The second-order polynomials pose a problem that they will not give accurate approximation and yield discontinuity when considering the second derivative of bond functions. Moreover, according to Martellini, Priaulet and Priaulet (2003), higher orders impose greater complexity "with no real justification about the continuity of the third or fourth derivative."

The third-order splines are generally called cubic splines. They are twice differentiable at all points on their domain. To have a smooth, fitted curve, at each knot points the following two conditions must hold. First, slopes on each side of every knot point must be equal. In addition, the curvature from each side must match.

Cubic polynomials can support a variety of yield-curve shapes between two particular knot points, with the possibility of up to one hump or one trough in each interval. The rest of this chapter is devoted to the basic spline theory.

Spline Interpolation

In numerical analysis, a spline is a special function defined piecewise by polynomials. Spline also refers to a wide range of functions that are used in applications requiring data interpolation and/or smoothing. A spline function consists of polynomial pieces on subintervals joined together with certain continuity conditions. The points where piecewise polynomials of adjacent subintervals meet are called knot points, or knots. Linkage of subintervals forms approximation space. In general, for n subintervals, there are $n+1$ knot points specified as $\{t_0,...,t_n\}, t_i < t_{i+1}, i = 0,...,n-1$. A spline approximation function with polynomial of degree k is called a k^{th}-order spline. Having knots $t_0,...,t_n$, a k^{th}-order spline function, *S*, satisfies the following properties. First, $S(t), t \in [t_0, t_n]$ is a polynomial of degree $\leq k$ on each interval $[t_i, t_{i+1}), i = 0,...,n-1$. Second, $S(t)$ has a continuous *k*-1$^{\text{st}}$ derivative on $[t_i, t_n]$, meaning that $S(t)$ is *k*-1 times differentiable everywhere on approximation space. Thus, $S(t)$ is a piecewise polynomial of degree at most k having continuous derivatives of all order up to *k*-1. For example, a spline of order three or a cubic spline *k*=3 is approximated with a piecewise cubic polynomial. A cubic spline is twice differentiable. Moreover, not only must slopes on each side of each knot point match, curvature from each side must also match.

Splines of Different Forms

Constant ($k = 0$)

Splines of order 0 are represented by piecewise polynomials of degree 0, which are constants. A spline of order 0 can explicitly be characterized in the form

$$S^0(t) = \begin{cases} S_0(t) = a_0 & t \in [t_0, t_1) \\ S_1(t) = a_1 & t \in [t_1, t_2) \\ S_2(t) = a_2 & t \in [t_2, t_3) \\ \vdots & \vdots \\ S_{n-1}(t) = a_{n-1} & t \in [t_{n-1}, t_n) \end{cases} \tag{16}$$

Linear ($k = 1$)

Splines of order 1 is represented by linear functions of degree 1 in the form

$$S^1(t) = \begin{cases} S_0(t) = a_0 + b_0 t & t \in [t_0, t_1) \\ S_1(t) = a_1 + b_1 t & t \in [t_1, t_2) \\ \vdots & \vdots \\ S_{n-1}(t) = a_{n-1} + b_{n-1} t & t \in [t_{n-1}, t_n) \end{cases} \tag{17}$$

Linear splines must satisfy a continuity condition such that $S_{i-1}(t_i) = S_i(t_i),\ 1 \le i \le n-1$.

Quadratic ($k = 2$)

Quadratic splines not only require continuity of spline functions as do the linear splines, but they also impose differentiability and continuation of first derivative at knot t_i to guarantee a smooth curve. This implies that the first derivative of S at t exists, which rules out a corner, a cusp and a vertical tangent. There must also be no discontinuity in the first derivative at knots t_i. That is $S_{i-1}(t_i) = S_i(t_i)$ and $S'_{i-1}(t_i) = S'_i(t_i),\ 1 \le i \le n-1$

$$S^2(t) = \begin{cases} S_0(t) = a_0 + b_0 t + c_0 t^2 & t \in [t_0, t_1) \\ S_1(t) = a_1 + b_1 t + c_1 t^2 & t \in [t_1, t_2) \\ \vdots & \vdots \\ S_{n-1}(t) = a_{n-1} + b_{n-1} t + c_{n-1} t^2 & t \in [t_{n-1}, t_n) \end{cases} \tag{18}$$

Cubic ($k = 3$)

Similar to the quadratic splines, cubic splines have continuity condition of their functions and first derivative of the functions. In addition, the curvature on both sides of each knot must equal so that $S_{i-1}(t_i) = S_i(t_i)$, $S'_{i-1}(t_i) = S'_i(t_i),\ 1 \le i \le n-1$ and $S''_{i-1}(t_i) = S''_i(t_i),\ 1 \le i \le n-1$

$$S^3(t) = \begin{cases} S_0(t) = a_0 + b_0 t + c_0 t^2 + d_0 t^3 & t \in [t_0, t_1) \\ S_1(t) = a_1 + b_1 t + c_1 t^2 + d_1 t^3 & t \in [t_1, t_2) \\ \vdots & \vdots \\ S_{n-1}(t) = a_{n-1} + b_{n-1} t + c_{n-1} t^2 + d_{n-1} t^3 & t \in [t_{n-1}, t_n) \end{cases} \tag{19}$$

A general cubic spline, equivalent to that used by McCulloch(1971), can be written in a spline functional form composed of power functions and truncated power functions.

$$S(t) = \sum_{l=0}^{3} a_l t^l + \frac{1}{3!}\sum_{i=1}^{n-1} b_i (t - t_i)_+^3 \quad (20)$$

where $(t - t_i)_+ = \max(0,\ t - t_i)$. The first part of the formula is represented by the third-degree power functions or cubic polynomials. The second part, where the truncated power functions lie, is twice differentiable at knot points. Both power functions $\{t^l\}_{l=0,\ldots,3}$ and truncated power functions $\{(t - t_i)_+^3\}_{i=1,\ldots,n-1}$ form a set of basis functions. These functions span approximation space of *S*(*t*). Alternatively, *S*(*t*) is a linear combination of linearly-dependent basis functions. For a cubic spline *(k=3)* with n subinterval or $n+1$ knots, $\{t_0,\ldots,t_n\}, t_i < t_{i+1}, i = 0,\ldots,n-1$, there are *n+3* parameters that has to be estimated. These are $\{a_0,\ldots a_3, b_1,\ldots,b_{n-1}\}$. In addition, for a k^{th}-order spline, it is apparent that $n+k$ parameters corresponding to the basis functions are $\{a_0,\ldots a_k, b_1,\ldots,b_{n-1}\}$.

A major drawback of the specification of *S*(*t*) in Equation (20), as shown by Powell (1981), is that not all bases are equally useful in defining spline regressors that provide reliable estimation. The second term in Equation (20) can give large negative numbers especially at high maturity leading to inaccuracy. Moreover, some bases result in regressor matrix that are almost perfectly collinear. This problem is due to the fact that the basis functions $\{t^l\}_{l=0,\ldots,3}$ and $\{(t - t_i)_+^3\}_{i=1,\ldots,n-1}$ are unbounded. The yield curve modeling and computation can be made easy and stable by defining basis functions that are bounded. General splines can then be formulated as a linear combination of a set of basis functions, thus the name B-splines.

B-Splines

Spline functions can be formed through linear combination into a system of spline functions. Splines that provide bases for, or span, certain spline spaces are called B-splines. For fixed knots t_i on the real line, where $\ldots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \ldots$ and $t \in \mathbb{R}$, B-spline can be generated by recurrence relations. Higher-order B-splines generated from iteration of lower-order B-splines preserve their properties. According to Kincade and Cheney (2002), the basic recurrence relation is

$$B_i^k(t) = \left(\frac{t - t_i}{t_{i+k} - t_i} \right) B_i^{k-1}(t) + \left(\frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} \right) B_{i+1}^{k-1}(t) \qquad \text{, for } k \geq 1 \tag{21}$$

The following discusses the B-splines of degree zero, the B-splines of degree zero and their recursive nature.

B-Spline of Degree Zero

B-Splines of degree zero are the most fundamental of type B-splines. Unlike the general splines of order zero defined in the previous section, B-Splines of degree zero are bounded.

$$B_i^0(t) = \begin{cases} 1 & , \ t_i \leq t < t_{i+1} \\ 0 & , \ otherwise \end{cases} \tag{22}$$

The value of $B_i^0(t)$ at t is one for $t \in [t_i, t_{i+1}), i \in \mathbb{Z}$, where set $\mathbb{Z}$ composes of all integer values, and zero outside $[t_i, t_{i+1})$. $[t_i, t_{i+1}]$ is called the support of $B_i^0(t)$. $B_i^0(t)$ has the following important properties.

1. $B_i^0(t) \geq 0$ for all i and all t.
2. B_i^0 is right continuous on the entire real line.
3. $\sum_{i=-\infty}^{\infty} B_i^0(t) = 1$ for all t.

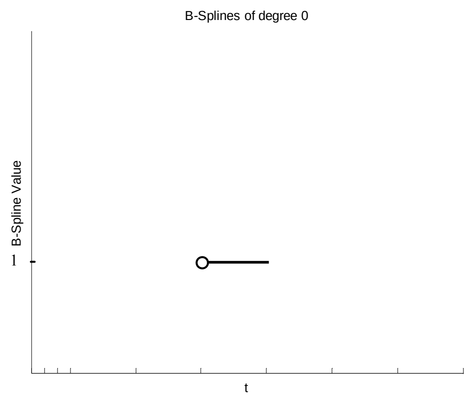

Figure 1. The B-spline B_i^0 .

B-Spline of Degree One

From Equation (21), B_i^1 can be written as a linear combination of two B-splines of degree zero as

$$B_i^1(t) = \left(\frac{t - t_i}{t_{i+1} - t_i}\right) B_i^0(t) + \left(\frac{t_{i+2} - t}{t_{i+2} - t_{i+1}}\right) B_{i+1}^0(t) \tag{23}$$

$$B_i^1(t) = \begin{cases} 0 & , \ t < t_i \quad or \quad t \geq t_{i+2} \\ \dfrac{t - t_i}{t_{i+1} - t_i} & , \ t_i \leq t < t_{i+1} \\ \dfrac{t_{i+2} - t}{t_{i+2} - t_{i+1}} & , \ t_{i+1} \leq t < t_{i+2} \end{cases} \tag{24}$$

The support of B_i^1 is $[t_i, t_{i+2})$. Similar properties hold for B_i^1.

1. $B_i^1(t) \geq 0$ for all i and all t.

2. B_i^1 is continuous and differentiable everywhere with an exception at knots t_i, t_{i+1} and t_{i+2}.

3. $\sum_{i=-\infty}^{\infty} B_i^1(t) = 1$ for all t.

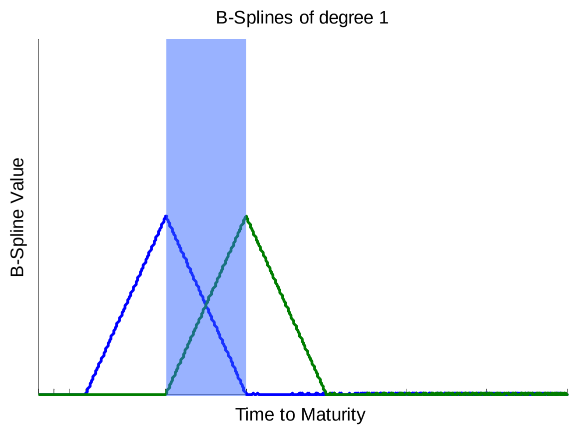

Figure 2. The B-spline B_i^1.

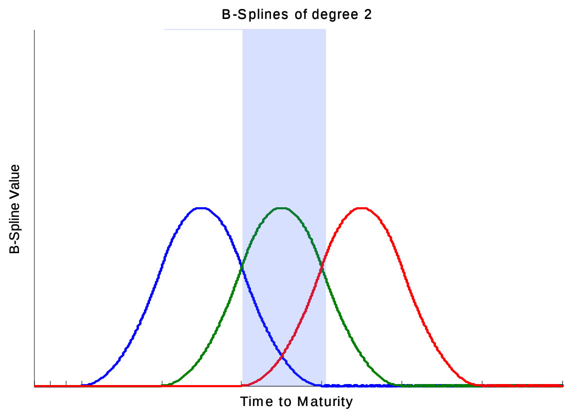

Figure 3. The B-spline B_i^2 .

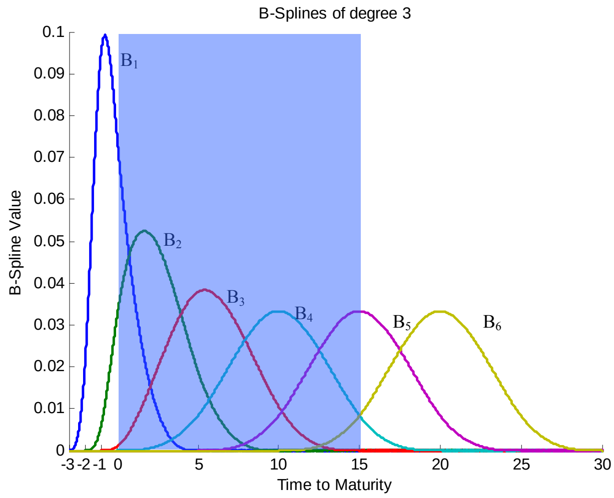

Figure 4. The B-spline B_i^3 .

B-Spline of Higher Orders

Higher-order B-spline functions can be derived from the recurrence method described above. B_i^1 is derived from B_i^0 and B_{i+1}^0. $B_i^2(t)$ is derived from B_i^1 and B_{i+1}^1. Therefore, $B_i^2(t)$ is composed of B_i^0, B_{i+1}^0 and B_{i+2}^0. Deduction of $B_i^3(t)$ in the same manner yield the components B_i^0, B_{i+1}^0, B_{i+2}^0 and B_{i+3}^0. Given this reasoning, explicit functional forms of k^{th}-order B-splines are easily obtained.

In the literature, another formula proposed by Powell (1981) provides a practical form of B-splines given by

$$B_p^k(t) = \sum_{i=p}^{p+k+1} \left[\prod_{j=p, j \neq i}^{p+k+1} \frac{1}{\left(t_j - t_i\right)} \right] \left(t - t_i\right)_+^k \qquad (25)$$

where $(t - t_i)_+^k$ is the truncated power function which is equivalent to $\left[\max(0,\ t - t_i)\right]^k$. With this B-spline specification the set of basis functions are not normalized. It can be proven that there is *t* such that the property $\sum_{i=-\infty}^{\infty} B_i^k(t) = 1$ does not hold. Nevertheless, the non-uniformity of basis functions, evident from Steeley (1991), Equation (25) does not affect the validity of this method.

Curve Fitting Using a Basis

In a linear family of curves, function *S*(*t*) is approximated by a linear combination of a set of spline basis functions $B_p(t)$.

$$S(t) = \sum_{p=1}^{P} \lambda_p B_p(t) \qquad (26)$$

where λ_p are coefficients corresponding to the p^{th}-spline that determines *S*(*t*). Let $\lambda_{p\times 1} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{bmatrix}$ and $B_{p\times h} = \begin{bmatrix} B_1(t_1) & \cdots & B_1(t_h) \\ \vdots & \ddots & \vdots \\ B_p(t_1) & \cdots & B_p(t_h) \end{bmatrix}$. Thus, in the matrix form $S = B'\lambda$.

The following notations are defined for bond pricing. Let Q_u represent the price of bond *u* and $Q = \begin{bmatrix} Q_1 \\ \vdots \\ Q_u \end{bmatrix}$. Matrix *C* of dimensions $u \times h$ contains cashflows from bond *u* paid at time t_m, *m=1,...,h*. c_{um} is zero if bond *u* does not deliver cash payout (interest and/or principle) at time t_m. The curve *S*(*t*), approximated from B-spline basis functions, is then used in bond pricing equation as follows.

$$Q = CS + \varepsilon \qquad (27)$$

Substituting $S = B'\lambda$ into Equation (27) yields

$$Q = CB'\lambda + \varepsilon \qquad (28)$$

Let $D = CB'$, Equation (28) can be rewritten as

$$Q = D\lambda + \varepsilon \tag{29}$$

It is then straight forward to find λ that satisfies Equation (29) by the least square criterion.

Chapter 4

Data and Methodology

In finance and economics, yield curves represent a relationship between yield to maturity and term of fixed income securities at a particular time. The yield curve and the term structure of interest rates are sometimes used interchangeably. There is no agreement in literature on the distinction between both terms. While the notion of yield curve is attached to graphical depiction, the term structure of interest rates is a broader concept which could be the discount function, the spot function or other price or yield relationship between fixed-income securities. Nevertheless, the term structure of interest rates in the modeling context is generally confined to zero-coupon rates only. In most markets, traded government bonds include both discount bonds and coupon bonds. Very short-maturity bonds of one year or less are usually traded on a discount basis. Even though some zero-coupon bonds are available for longer maturities, coupon bonds constitute most of the longer maturities. Hence, with a mixture of both types of bonds, the term structure of interest rates that builds up zero-coupon yield curves is not directly observable. To obtain the term structure in light of the issue discussed, econometric and statistical methods are required.

Data

Since the term structure requires short-term, medium-term and long-term government fixed income instruments, yield on the treasury bills and government bonds with maturity ranging from 1 year to 15 years are required. This study investigates different fitting models using the Thai Bond Market Association (ThaiBMA) data on Friday, January 13, 2006. There are 12 treasury bills (TB series) and 28 government bonds (LB series) outstanding on the specified date. The total sample size is 40. Table 1 displays all the LB-series Thai government bonds outstanding on January 13, 2006.

Table 1
ThaiBMA Registered Government Bonds as of January 13, 2006

ThaiBMA Symbol	Payment Date (M/D)	Coupon Type	Coupon Rate (%)	Issue Date (M/D/Y)	Maturity Date (M/D/Y)	Issue Term (years)	Issue Size (M. THB)
LB061A	1/24, 7/24	Fixed	2	1/24/2003	1/24/2006	3	15,000.00
LB06DA	6/8, 12/8	Fixed	8	12/8/1998	12/8/2006	8	35,000.00
LB077A	1/7, 7/7	Fixed	5.6	7/7/2000	7/7/2007	7	37,000.00
LB082A	2/12, 8/12	Fixed	4.125	2/12/2001	2/12/2008	7	34,950.00
LB085A	5/14, 11/14	Fixed	2.75	5/14/2004	5/14/2008	4	15,000.00
LB088A	2/5, 8/5	Fixed	3.875	8/5/2005	8/5/2008	3	37,500.00
LB08DA	6/8, 12/8	Fixed	8.5	12/8/1998	12/8/2008	10	50,000.00
LB096A	6/21, 12/21	Fixed	4.625	6/21/2002	6/21/2009	7	40,000.00
LB09NC	5/19, 11/19	Fixed	4.125	11/19/2004	11/19/2009	5	40,000.00
LB104A	4/9 10/9	Fixed	4.8	4/9/2001	4/9/2010	9	39,440.00
LB108A	2/13, 8/13	Fixed	4.25	8/13/2004	8/13/2010	6	30,000.00
LB111A	1/9, 7/9	Fixed	3.875	1/9/2004	1/9/2011	7	30,000.00
LB113A	3/5, 9/5	Fixed	7.5	3/5/1999	3/5/2011	12	39,836.00
LB11NA	5/30, 11/30	Fixed	5.375	11/30/2001	11/30/2011	10	39,100.00
LB123A	3/11, 9/11	Fixed	4.5	3/11/2005	3/11/2012	7	40,000.00
LB12NA	5/1, 11/1	Fixed	4.125	11/1/2002	11/1/2012	10	35,000.00
LB137A	1/13, 7/13	Fixed	5.25	7/13/2006	7/13/2013	7.5	20,000.00
LB13OA	4/17, 10/17	Fixed	4	10/17/2003	10/17/2013	10	25,000.00
LB143A	3/5, 9/5	Fixed	8.25	3/5/2000	3/5/2014	15	40,000.00
LB145A	5/14, 11/14	Fixed	4.875	5/14/2004	5/14/2014	10	10,000.00
LB14DA	6/3, 12/3	Fixed	5	12/3/2004	12/3/2014	10	32,000.00
LB157A	1/7, 7/7	Fixed	7.2	7/7/2000	7/7/2015	15	35,950.00
LB171A	1/18, 7/18	Fixed	5.5	1/18/2002	1/18/2017	15	40,000.00
LB183A	3/7, 9/7	Fixed	3.875	3/7/2003	3/7/2018	15	25,000.00
LB198A	2/13, 8/13	Fixed	5.5	8/13/2004	8/13/2019	15	34,800.00
LB19DA	6/3, 12/3	Fixed	5.375	12/3/2004	12/3/2019	15	36,900.00
LB214A	4/9, 10/9	Fixed	6.4	4/9/2001	4/9/2021	20	40,000.00
LB22NA	5/8, 11/8	Fixed	5.125	11/8/2002	11/8/2022	20	26,000.00

Note. Source: The Thai Bond Market Association

The Term Structure Fitting Using B-Splines

From the previous chapter, Equation (25) provides a set of k^{th}-order, p^{th} B-splines for fixed knot points t_i, where $\ldots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \ldots$. To be certain that the condition for continuity of the second derivative is met, this study implements the cubic B-splines. Since cubic B-splines are of order $k=3$, Equation (25) becomes

$$B_p^3(t) = \sum_{i=p}^{p+4} \left[\prod_{j=p,j\neq i}^{p+4} \frac{1}{\left(t_j - t_i\right)} \right] \left(t - t_i\right)_+^3 \tag{30}$$

In addition, other information has to be defined, namely the number of subintervals (n) in the approximation space. Subintervals determine the number of knot points required in estimation. For n subinterval, n+1 knots are needed to completely cover the approximation space. These n+1 knots are called in-sample knots. Not only must the in-sample knots be specified, out-of-sample knots are also need to reduce the bias at both ends of the approximation space by increasing the number of bases on a first and last subinterval in the approximation space. The number of extra knots added to each end is k, the order of B-splines. Hence, the total number of knots used in yield curve estimation is n+1+2k.

Moreover, given the number of subintervals n and order k, the number of basis functions is $n+k$. For example, for B-splines of order 3 on an approximation space that contains 3 subintervals, a total of 6 basis functions and 10 knots is present. Also, the value of basis function is positive on $\left(t_i, t_{i+k+1}\right)$, and otherwise zero. For B-splines of order 3, this interval is $\left(t_i, t_{i+4}\right)$. Finally, inside the approximation space, there are k+1 basis functions and the number of these functions reduces by 1 on each subinterval further away from the space. For $k=3$, there are 4 basis functions on each subinterval in the space. The number of basis functions is 3 for the subinterval adjacent to the approximation space, and declines to 2 and 1 when moving away each subinterval from the space.

In this study, the approximation space is $[t_0, t_n]$, containing knot points $\left\{t_i\right\}_{i=0,\ldots,n}$. Also, 6 out-of-sample knot points $t_{-3}, t_{-2}, t_{-1}, t_{n+1}, t_{n+2}, t_{n+3}$ are added to complete the set of knots. Once all knot points are set as $t_{-3} < t_{-2} < t_{-1} < t_0 < \ldots < t_n < t_{n+1} < t_{n+2} < t_{n+3}$, all n+3 spline basis functions $\left\{B_p(t)\right\}_{p=-3,\ldots,n+3}$ are defined on approximation space. The B-spline approximating function on $[t_0, t_n]$ is

$$S(t) = S\left(t \middle| \lambda_1, \ldots, \lambda_{n+3}\right) = \sum_{p=1}^{n+3} \lambda_p B_p(t) \qquad (31)$$

In matrix form, $S = B'\lambda$. In the term structure of interest rate estimation, matrix S is sought in such a way that it minimizes pricing error. Pricing error is defined as the difference between the quoted market price of bonds and the theoretical price estimated from cashflow streams using B-spline approximation. Thus, Equation (27) can be rewritten as

$$\varepsilon = Q - CS \tag{32}$$

Through a 2-step transformation by $S = B'\lambda$ and $CB' = D$ described at the end of Chapter 3, a regression analysis under the minimum residual sum of squares criterion can be applied to derive the coefficients of B-spline basis function, λ's, such that λ satisfies the following condition

$$\lambda^* = \arg\min_{\lambda} \left\{ \varepsilon'\varepsilon \middle| \varepsilon = Q - D\lambda \right\} \tag{33}$$

The Term Structure Fitting Methodologies

The fundamental concept of bond price states that the price of a bond is the sum of all the present value of coupon payments and principal over the bond's life. Alternatively, it is the sum of all future cashflows discounted at prevailing interest rates. In a series of cashflows of a coupon bond, each cash payment can be views as a strip of a zero-coupon bond. Assuming that there are n default-free coupon bonds in the sample, the price of the coupon bond u is a linear combination of a series of pure discount bond prices. This relationship can be represented by

$$Q_u = \sum_{m=1}^{h_u} y_u(t_m) P(t_m) \ , \qquad u = 1, 2, \ldots, v \tag{34}$$

where t_m is the time when the m^{th} coupon or principal payment is made. h_u is the total number of coupon payments of bond u. $y(t_m)$ is the cashflow paid by bond u at time t_m. $P(t_m)$ is the pure discount bond price with a face value of 1. $P(t_m)$ can also be regarded as a discount factor.

According to Weierstrass Approximation Theroem (Powell 1981), a continuous function can be arbitrary closely approximated by a set of functions. A spline function approximation technique is applied to estimate bond price. Because a spline is a piecewise polynomial approximation using different sets of functions over a domain, the approximation function for bond price should be dependent on time to maturity.

Model Formulation

This section incorporates the spot rate, forward rate, and discount function from the earlier sections to form the price function for coupon bond. The following formulation adjustments are needed. First, all time is denoted by *t*. $P(t)$ is the price at time *t* of a zero-coupon bond. $r(t)$ is the spot interest rate or the pure discount bond yield at time *t*. $f(t)$ is the instantaneous forward rate at time *t*. Redefining the spot rate, Equations (8) and (14) are simplified to

$$r(t) = -\frac{\ln P(t)}{t} \tag{35}$$

$$P(t) = e^{-r(t)t} \tag{36}$$

and the forward rate from Equations (9) and (15) are simplified to

$$f(t) = -\frac{\partial}{\partial t}\ln P(t) \tag{37}$$

$$P(t) = e^{-\int_0^t f(s)ds} \tag{38}$$

Discount Fitting Model

Discount function is approximated by using B-splines by

$$P(t) = \sum_{p=1}^{n+3} \lambda_p B_p(t) \tag{39}$$

where $B_p(t)$ is the p^{th} approximation function derived using B-spline approximation, λ_p's are coefficients that need to be estimated for p^{th} approximation function.

Combining Equations (34) and (39), the bond price estimated by the discount fitting model with an error term added is then

$$Q_u = \sum_{p=1}^{n+3} \lambda_p \left(\sum_{m=1}^{h_u} y_u(t_m) B_p(t_m) \right) + \varepsilon_u \tag{40}$$

An additional restrictive condition can be placed on the discount fitting model. In this model $P(t)$ is the discount factor used to discount coupon and principle payments. It is customary that $P(0)$, the discount factor at time $t = 0$, is equal to 1.

The value of payments at the present time needs no discounting. Therefore, this restriction can be expressed as

$$P(0) = \sum_{p=1}^{n+3} \lambda_p B_p(0) = 1 \tag{41}$$

Spot Fitting Model

Spot function is approximated by using B-splines by

$$r(t) = \sum_{p=1}^{n+3} \lambda_p B_p(t) \tag{42}$$

The result from combining Equations (34), (36) and (42), and adding an error term yields

$$Q_u = \sum_{m=1}^{h_u} y_u(t_m) * \exp\left[-t_m \sum_{p=1}^{n+3} \lambda_p B_p(t_m)\right] + \varepsilon_u \tag{43}$$

Forward Fitting Model

Forward function is approximated by using B-splines by

$$f(t) = \sum_{p=1}^{n+3} \lambda_p B_p(t) \tag{44}$$

The result from combining Equations (34), (38) and (44), and adding an error term yields

$$Q_u = \sum_{m=1}^{h_u} y_u(t_m) * \exp\left[-\int_0^{t_m} \left(\sum_{p=1}^{n+3} \lambda_p B_p(s)\right) ds\right] + \varepsilon_u \tag{45}$$

Model Comparison

The three models, discount fitting, spot fitting and forward fitting are then evaluated by regression analysis using the least square method. All the coefficients λ_p's of each model are estimated. These coefficients are the foundation for calculating the theoretical bond price and zero-coupon yield at any time to maturity, between zero to fifteen years in this study. Furthermore, the results of each model vary due to three important determinants. These are parameters employed in the estimation, namely the degree of polynomial (*k*), the number of interval within the estimation horizon (*n*), and the location of knots. Spline bases utilized in approximation are formed from polynomials of degree k = 3 and 4 and intervals of n =

1, 2, 3 and 4. As previously mentioned, the number of degrees and intervals gives a rule for the required number of knot points. However, the knot location can be chosen arbitrary. To make it systematic, the within-sample knots (knots that are in between 0 and 15) are only defined by integers. For example, there are 1 knot combination for n = 1, 14 knot combinations for n = 2, 91 knot combinations for n = 3, and the like.

To find the optimal set of model determinants, various degrees of polynomial, numbers of approximation interval and possible knot locations are matched and estimated. Two error criteria implemented in this study is the generalized cross validation (GCV) and the mean integrated squared error (MISE). These are among indices that are useful in evaluation of regression equation.

The generalized cross validation (GCV) was developed to improve the feature of cross validation (CV). As stated by Eubank (1999), generalized cross validation is based on residual sum of squares which is defined as

$$GCV = \frac{RSS / m}{\left(1 - \frac{k+1+n}{m}\right)^2} \tag{46}$$

where RSS is residual sum of squares

k is the degree of B-spline polynomials such that k = 2, 3, 4, …

n is the number of approximation intervals such that n = 1, 2, 3, …

m is sample size

The minimum GCV value indicates the condition of optimal knot position, degree of polynomials, and the number of approximation intervals.

The comparisons are also made with the data of zero-coupon yield reported by the ThaiBMA. The criterion used is the mean integrated squared error (MISE). MISE requires the practical values of the function to be known. Without any adjustment, MISE is useful when the distribution of the predicted value is close to the predictor in practical applications (Takezawa 2006).

$$MISE = \frac{1}{15}\int_0^{15} \left(\hat{f}(t) - f(t)\right)^2 dt \tag{47}$$

where $\hat{f}(t)$ is the yield curve derived from the B-spline approximation

$f(t)$ is the ThaiBMA interpolated zero-coupon yield curve

MISE indicates how close the fitted curve is in comparison with the true curve.

The functional form of the ThaiBMA yield curve is unavailable. However, the ThaiBMA zero-coupon yield can be estimated from linear interpolation of the yield data. As a result, MISE can be closely approximated by

$$\widehat{MISE} = \frac{1}{15}\sum_{i=1}^{n}\left(\hat{f}(t_i) - f(t_i)\right)^2 \Delta t_i \quad (48)$$

where $\hat{f}(t_i)$ and $f(t_i)$ are the value at t_i of yields approximated from B-splines and linear interpolation respectively.

n is the number of intervals between 0 and 15.

MISE is the indicator for comparison among models with different B-spline determinants. The lower the MISE, the better approximation of the zero-coupon yield curve compared to the ThaiBMA yield curve. The term structure estimated from the superior model and determinants can be used as benchmark interest rates for the bond market.

Chapter 5

Empirical Results

Chapter 4 introduces the estimation methods and all four fitting models: the non-restricted discount fitting, the restricted discount fitting, the spot fitting and the forward fitting. All models are estimated with different degrees of B-spline polynomials (k), number of intervals within the estimation horizon (n), and many possible within-sample knot locations. For n greater than two, there are more than one possible knot combinations. This chapter discusses the result findings in order of selection criteria: GCV, MISE, and makes a comparison to the interpolated zero-coupon yield curve reported by the ThaiBMA.

Generalized Cross Validation (GCV)

Table 2 reports the minimum values of generalized cross validation (GCV) corresponding to the degrees of polynomial (k) and the number of intervals (n). For the linear B-splines, the minimum GCV values vary considerably across the number of estimation intervals, with an exception of the forward fitting model. In comparison to linear B-splines, quadratic B-splines provide much lower values of GCV. Nonetheless, the theoretical foundation invalidates order-one and order-two splines on the term structure application as the yield curve must be twice differentiable. On the other hand, as Martellini et al. (2003) pointed out, the splines of degrees greater than three places sophistication on the estimations with no apparent use of the continuity of the third- or higher-order derivatives. In fact, the GCV values of the quartic splines are not necessarily lower than those of the cubic splines. Therefore, as the cubic B-splines are preferred from the previously mentioned reasons, it is apparent from Table 2 that the minimum GCV across the fitting models are 0.6166. This matches with the cubic B-splines of two intervals at the in-sample knot location [0 3 15] in Table 3.

It can be noted that, for the optimal knot position, first within-sample knot is three, which is close to zero. In fact, it is can be observable that most of the first within-sample knots of cubic B-splines have value of at most three. The argument in support of this finding is that these period of short maturities consists of all treasury bills and coupon payment are distributed from almost all bonds. The first within-sample knot is placed to the left of approximation interval for greater accuracy.

Table 2
Minimum Values of Generalized Cross Validation (GCV)

Fitting Models	No. of intervals	Degrees of B-splines	
		3	4
Non-restricted discount fitting	1	0.6398	0.6571
	2	0.6504	0.6397
	3	0.6402	0.6764
	4	0.6720	0.7101
	5	0.6788	0.7080
Restricted discount fitting	1	0.6482	0.6780
	2	0.6669	0.6521
	3	0.6485	0.6910
	4	0.6812	0.7252
	5	0.6901	0.7237
Spot fitting	1	0.7835	24.9737
	2	**0.6166**	0.6506
	3	0.6482	0.6861
	4	0.6638	0.6909
	5	0.6722	0.7161
Forward fitting	1	1.2558	3.1312
	2	0.6502	0.6560
	3	0.6519	0.6910
	4	0.6849	0.7163
	5	0.6925	0.7430

Table 3
In-sample Knot Locations Corresponding to the Minimum Values of Generalized Cross Validation (GCV)

Fitting Models	No. of intervals	Degrees of B-splines	
		3	4
Non-restricted discount fitting	1	0 15	0 15
	2	0 9 15	0 1 15
	3	0 1 8 15	0 1 11 15
	4	0 1 10 11 15	0 1 13 14 15
	5	0 9 10 11 12 15	0 3 4 5 6 15
Restricted discount fitting	1	0 15	0 15
	2	0 1 15	0 1 15
	3	0 1 14 15	0 3 5 15
	4	0 1 10 11 15	0 2 13 14 15
	5	0 9 10 11 12 15	0 3 4 5 7 15
Spot fitting	1	0 15	0 15
	2	0 3 15	0 3 15
	3	**0 3 7 15**	0 2 10 15
	4	0 3 5 6 15	0 4 5 7 15
	5	0 3 4 10 11 15	0 3 4 5 6 15
Forward fitting	1	0 15	0 15
	2	0 2 15	0 3 15
	3	0 1 5 15	0 2 10 15
	4	0 9 11 12 15	0 1 13 14 15
	5	0 3 5 6 7 15	0 2 3 5 6 15

Table 4 displays the estimated results of all four fitting model using cubic B-splines and two approximation intervals. The spot fitting model gives the minimum mean absolute error in price of 0.004916 of par value. The standard errors in price are relatively the same. Even though, the fitting errors are similar across models, in these particular cases however, there are notable differences in the term structures. As shown in Figure 5, the term structures resulting from all four models exhibit no significant differences at the maturities longer than four years. However, at short maturities, the restricted discount fitting model overstates the yields for Treasury bills and yields from the non-restricted discount fitting are unbounded below. The spot fitting model appears to give a reasonable estimate of the term structure.

Table 4
Results of Model Estimation under the GCV Criterion (k = 3, n = 2)

	Non-restricted discount fitting		Restricted discount fitting		Spot fitting		Forward fitting	
Coefficients	Coefficient estimated	Standard deviation	Coefficient estimated	Standard deviation	Coefficient estimated	Standard deviation	Coefficient estimated	Standard deviation
λ_1	46.17187	1.1640	20.9119	10.5659	**-9.9300**	**0.5426**	-6.7427	1.3728
λ_2	28.24700	0.3453	33.5065	0.2102	**1.6937**	**0.1416**	1.7145	0.4234
λ_3	19.06914	0.2291	21.7546	0.2013	**1.5484**	**0.0597**	1.5216	0.1956
λ_4	11.29920	0.3204	13.5150	0.3845	**1.8390**	**0.0558**	2.3275	0.1613
λ_5	8.05766	1.1845	8.5383	0.8324	**1.9793**	**0.3786**	1.4286	0.9888
Mean absolute error in price	0.005407		0.005413		**0.004916**		0.006113	
Standard Error in price	0.004267		0.004401		**0.004572**		0.004037	

Note. Sample size = 40 for each fitting model.

Non-restricted discount fitting model: $Q_u = \sum_{p=1}^{n+3} \lambda_p \left(\sum_{m=1}^{h_u} y_u(t_m) B_p(t_m) \right) + \varepsilon_u$

Restricted discount fitting model: $Q_u = \sum_{p=1}^{n+3} \lambda_p \left(\sum_{m=1}^{h_u} y_u(t_m) B_p(t_m) \right) + \varepsilon_u$; with $P(0) = \sum_{p=1}^{n+3} \lambda_p B_p(0) = 1$

Spot fitting model: $Q_u = \sum_{m=1}^{h_u} y_u(t_m) * \exp\left[-t_m \sum_{p=1}^{n+3} \lambda_p B_p(t_m) \right] + \varepsilon_u$

Forward fitting model: $Q_u = \sum_{m=1}^{h_u} y_u(t_m) * \exp\left[-\int_0^{t_m} \left(\sum_{p=1}^{n+3} \lambda_p B_p(s) \right) ds \right] + \varepsilon_u$

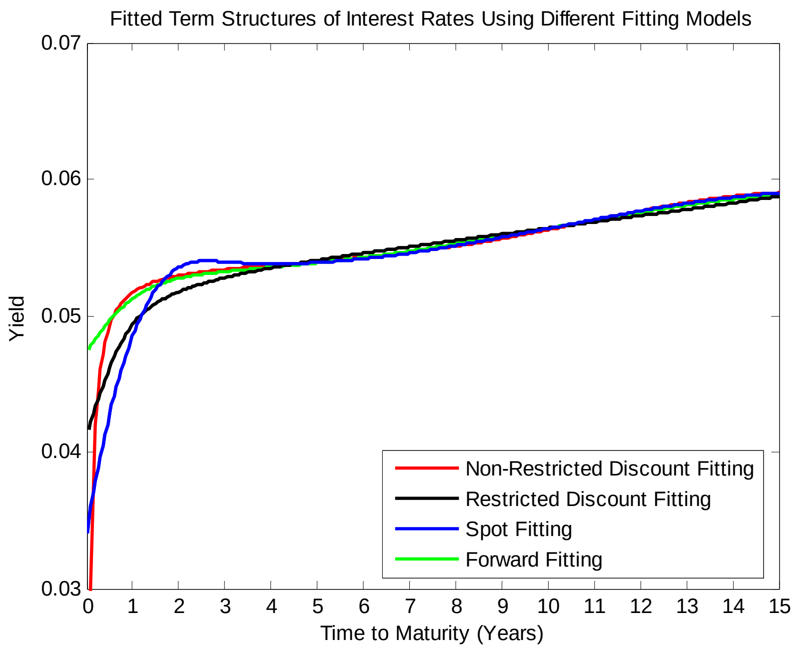

Figure 5. Fitted term structures of interest rates using different fitting models (k = 3, n = 2).

The hypothesis tests of the optimal case under the GCV criterion (spot fitting, k = 3, n = 2) are performed with the interval estimation of coefficients. The confidence intervals are derived from 600 iterations of bootstrap re-sampling. As shown in Table 5, all the estimated coefficients are statistically significant at 1% level. Increasing the number of iteration to 1000 or more does not drastically alter the width of confidence intervals.

Table 5

90%, 95% and 99% Confidence Intervals for Each Estimated Coefficient (Spot Fitting, k = 3, n = 2)

Coefficients	Coefficient estimated	Confidence interval					
		90% Confidence level		95% Confidence level		99% Confidence level	
		Lower	Upper	Lower	Upper	Lower	Upper
λ_1	-9.9300*	-10.8542	-9.0047	-10.9859	-8.8621	-11.4692	-8.5238
λ_2	1.6937*	1.4603	1.9410	1.4258	1.9878	1.3298	2.0746
λ_3	1.5484*	1.4471	1.6489	1.4290	1.6580	1.3923	1.6974
λ_4	1.8390*	1.7507	1.9370	1.7313	1.9559	1.7145	1.9966
λ_5	1.9793*	1.3817	2.6300	1.2961	2.7149	1.1116	2.8783

Note. * denotes statistical significance at 1% level.

The graphs of 90%, 95% and 99% confidence intervals plotted against the original estimated results of the spot fitting model in Table 5 are displayed in Figure 6. The shortest widths are approximately 1% and 1.5% at 90% confidence level and 95% confidence level respectively, occurring at intermediate maturities. At long maturities, however, the intervals tend to get wider. This is due to the fact that there are only a few government bonds with tenors longer than 12 years. Hence, the number of coupon and principle payments are very limited at the long horizon.

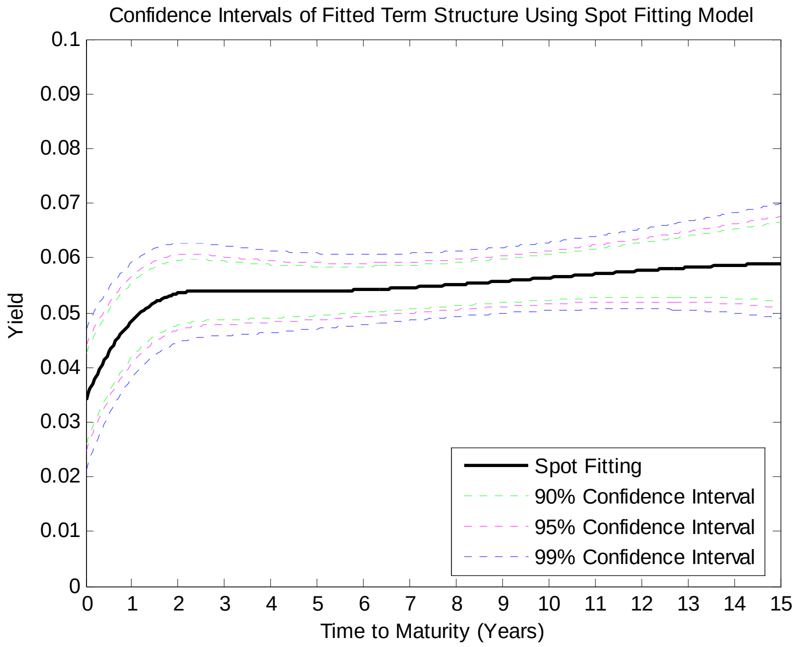

Figure 6. Fitted term structure of interest rates and its confidence intervals using spot fitting model (k = 3, n = 2).

Mean integrated squared error (MISE)

An alternative criterion for testing the goodness of fit of the fitting models is the mean integrated squared error (MISE) in comparison to the ThaiBMA interpolated zero-coupon yield curve. The results of the MISEs are reported in Table 6. In search of the minimum MISE in the similar manner as GCV, the minimum MISE found for cubic B-splines is 5.1625×10^{-6}. The optimal number of intervals for cubic B-splines is therefore three, under the restricted discount fitting model. Table 7 shows that the optimal in-sample knot location for the MISE criterion is at [0 1 14 15].

Table 8 displays the estimated results of all four fitting model using cubic B-splines and three approximation intervals. The mean absolute errors in price are approximately 0.5% and the standard errors in price range from 0.0044 to 0.0046. The differences are trivial and all term structure plots weave together in Figure 7. There are small discrepancies among the term structures from these models.

Table 6

Minimum Values of Mean Integrated Squared Error (MISE)

Fitting Models	No. of intervals	Degrees of B-splines	
		3	4
Non-restricted discount fitting	1	8.1659	12.5315
	2	7.1390	5.8159
	3	5.7929	5.8060
	4	5.7634	5.8075
Restricted discount fitting	1	11.6981	14.0616
	2	5.3741	5.2986
	3	**5.1625**	5.3254
	4	5.2161	5.3259
Spot fitting	1	28.5943	805.6486
	2	5.3937	5.1979
	3	5.3637	5.3449
	4	5.2896	5.2878
Forward fitting	1	54.6733	195.7474
	2	9.6238	5.3463
	3	5.2257	5.3309
	4	5.3070	5.3048

Note. All MISE values are reported as a multiple of 10^{-6}.

Table 7

In-sample Knot Locations Corresponding to the Minimum Values of Mean Integrated Squared Error (MISE)

Fitting Models	No. of intervals	Degrees of B-splines	
		3	4
Non-restricted discount fitting	1	0 15	0 15
	2	0 2 15	0 3 15
	3	0 3 5 15	0 4 6 15
	4	0 3 4 14 15	0 4 5 7 15
Restricted discount fitting	1	0 15	0 15
	2	0 2 15	0 2 15
	3	**0 1 8 15**	0 1 11 15
	4	0 1 13 14 15	0 3 10 14 15
Spot fitting	1	0 15	0 15
	2	0 4 15	0 8 15
	3	0 2 7 15	0 4 9 15
	4	0 3 7 10 15	0 4 10 14 15
Forward fitting	1	0 15	0 15
	2	0 1 15	0 3 15
	3	0 3 7 15	0 4 13 15
	4	0 1 2 6 15	0 2 3 7 15

Table 8
Results of Model Estimation under the MISE Criterion (k = 3, n = 3)

	Non-restricted discount fitting		Restricted discount fitting		Spot fitting		Forward fitting	
Coefficients	Coefficient estimated	Standard deviation	Coefficient estimated	Standard deviation	Coefficient estimated	Standard deviation	Coefficient estimated	Standard deviation
λ_1	20.3312	2.5008	**14.7421**	**3.7146**	-2.6445	0.3716	-4.8588	0.3406
λ_2	15.8184	0.2851	**25.8484**	**0.1514**	0.8933	0.1237	1.2564	0.1164
λ_3	17.1910	0.1052	**14.7316**	**0.1521**	1.0830	0.0462	1.0081	0.0428
λ_4	13.0600	0.1388	**11.2757**	**0.1820**	1.0809	0.0177	1.1851	0.0254
λ_5	9.5120	0.2569	**8.7929**	**0.2038**	1.3274	0.0528	1.6465	0.0551
λ_6	7.7745	0.4664	**5.6180**	**0.7136**	1.3475	0.2858	0.7464	0.1272
Mean absolute error in price	0.005227		**0.005010**		0.004859		0.004945	
Standard Error in price	0.004404		**0.004639**		0.004601		0.004578	

Note. Sample size = 40 for each fitting model.

Non-restricted discount fitting model: $Q_u = \sum_{p=1}^{n+3} \lambda_p \left(\sum_{m=1}^{h_u} y_u(t_m) B_p(t_m) \right) + \varepsilon_u$

Restricted discount fitting model: $Q_u = \sum_{p=1}^{n+3} \lambda_p \left(\sum_{m=1}^{h_u} y_u(t_m) B_p(t_m) \right) + \varepsilon_u$; with $P(0) = \sum_{p=1}^{n+3} \lambda_p B_p(0) = 1$

Spot fitting model: $Q_u = \sum_{m=1}^{h_u} y_u(t_m) * \exp\left[-t_m \sum_{p=1}^{n+3} \lambda_p B_p(t_m) \right] + \varepsilon_u$

Forward fitting model: $Q_u = \sum_{m=1}^{h_u} y_u(t_m) * \exp\left[-\int_0^{t_m} \left(\sum_{p=1}^{n+3} \lambda_p B_p(s) \right) ds \right] + \varepsilon_u$

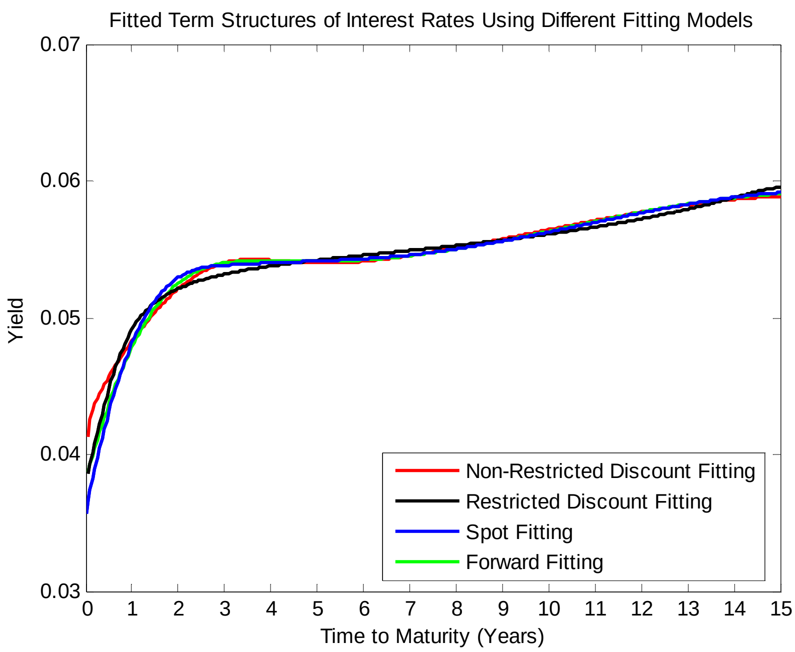

Figure 7. Fitted term structures of interest rates using different fitting models (k = 3, n = 3).

The confidence intervals of restricted discount fitting under MISE are shown in the same manner as those of GCV. From Table 9, all the estimated coefficients are significant at 1% level. In contrast to Figure 6, Figure 8 shows that the confidence intervals for short maturities are very large due to the fitting model. Nevertheless, these intervals get narrower as the time to maturity increases.

Table 9

90%, 95% and 99% Confidence Intervals for Each Estimated Coefficient (Restricted Discount Fitting, k = 3, n = 3)

Coefficients	Coefficient estimated	Confidence interval					
		90% Confidence level		95% Confidence level		99% Confidence level	
		Lower	Upper	Lower	Upper	Lower	Upper
λ_1	14.7421*	8.2782	20.4095	6.6548	21.7498	2.8722	22.9186
λ_2	25.8484*	25.5962	26.1076	25.5690	26.1544	25.4997	26.3236
λ_3	14.7316*	14.4600	14.9820	14.4220	15.0243	14.3556	15.1203
λ_4	11.2757*	10.9801	11.5577	10.9223	11.6237	10.7979	11.7636
λ_5	8.7929*	8.4503	9.1338	8.4089	9.2183	8.2517	9.4354
λ_6	5.6180*	4.5614	6.9330	4.3311	7.0500	3.6950	7.5496

Note. * denotes statistical significance at 1% level.

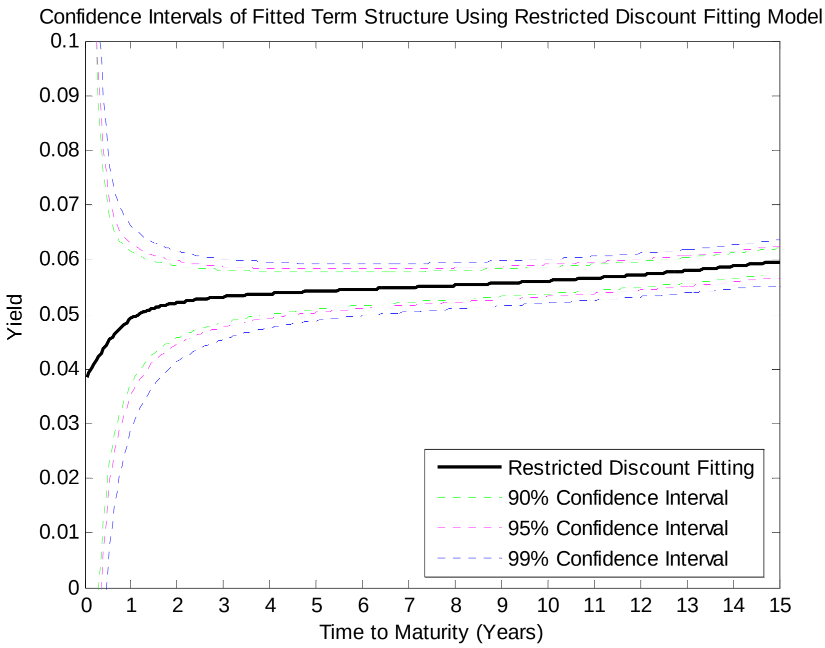

Figure 8. Fitted term structure of interest rates and its confidence intervals using restricted discount fitting model (k = 3, n = 3).

Comparison of GCV and MISE to the ThaiBMA

Figure 9 compares the optimal term structures derived from the GCV and MISE criteria together with the ThaiBMA interpolated zero-coupon yield curve. It is apparent that the two-interval cubic B-splines of the spot fitting model under the GCV criterion is similar to the three-interval cubic B-splines of the restricted discount fitting model. By the MISE criterion, the derived yield curve is the closet to the ThaiBMA yield curve. However, it intertwines the GCV curve and remains above that of the ThaiBMA almost in a parallel manner, especially at maturities of four years and greater. This is an indication this study applies a different data set in estimating the term structure than what the ThaiBMA uses. Consequently, these three yield curves displayed in Figure 9 are not directly comparable.

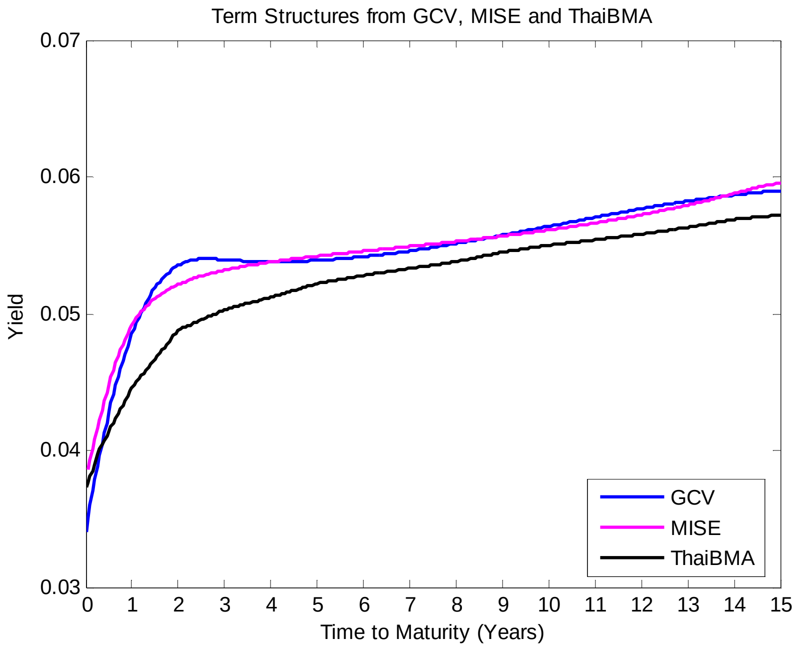

Figure 9. Fitted term structures of interest rates under the GCV and MISE criteria in comparison with the ThaiBMA yield curve.

However, Figure 10 and Figure 11 show that although the ThaiBMA yield curve lies almost parallel to the GCV and MISE curve, the ThaiBMA curve is still within the narrowest band of 90% confidence interval generated from the bootstrapping of GCV and MISE. This is an indication that the approximation of the term structure of interest rates by the B-spline approach is valid.

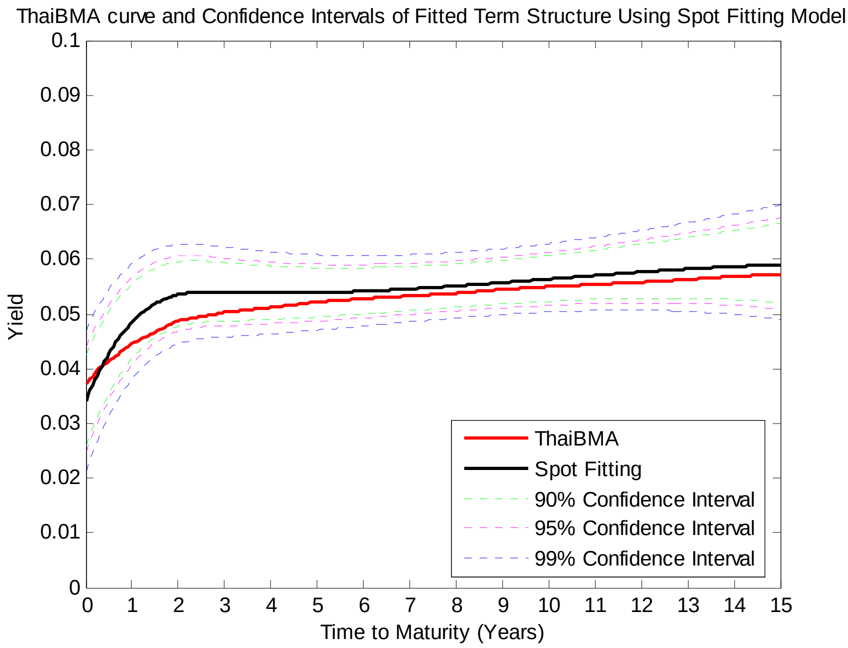

Figure 10. The ThaiBMA yield curve, fitted term structure of interest rates and its confidence intervals using restricted discount fitting model (k = 3, n = 3).

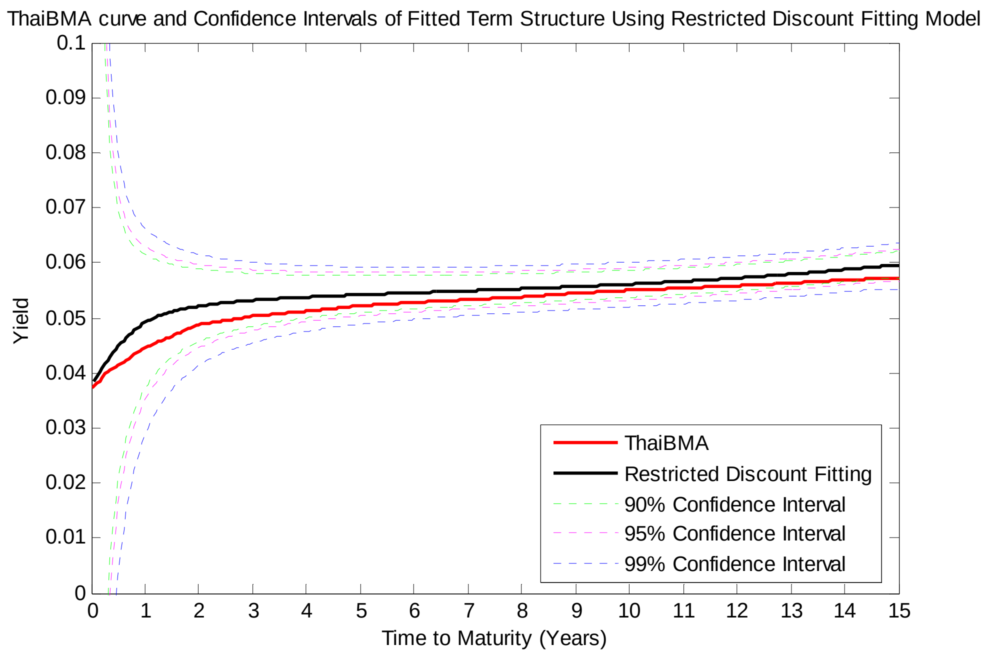

Figure 11. The ThaiBMA yield curve, fitted term structure of interest rates and its confidence intervals using spot fitting model (k = 3, n = 2).

Chapter 6

Conclusions

B-spline curve fitting is a global approximation technique which has many applications. The application of B-splines approximation to the term structure of interest rates has been explored since the 1970s. To the present time, its popularity has not declined. This study applies the B-spline as the empirical methodology for estimating the term structure of interest rates for Thai government bonds. B-splines are applied to four fitting models: non-restricted discount fitting, restricted discount fitting, spot fitting, and forward fitting.

One of the major requirements of B-spline application to the term structure is that knot points have to be pre-specified. The shapes of fitted curve are, therefore, dependent on knot location. To indicate the optimal knot location, this study addresses the method of generalized cross validation (GCV). With the bond trading data on Friday, January 13, 2006, the GCV indicates that, with the use of cubic B-splines, the spot fitting is better than both the non-restricted and forward fittings. Moreover, as the number of approximation intervals increases to more than three, GCV increases as well. Thus, this study shows that as the number of approximation interval becomes larger, the estimation result is not optimal.

The lowest GCV of cubic B-spline corresponds to spot fitting model with two approximation intervals. The first within-sample knot is close to zero. In fact, it is generally observed that, for cubic B-splines, the first within-sample knot has a value of three or lower. This reflects the need to support the estimation of short times to maturity during which coupons from most bonds are paid out.

A crucial remark should be made on the mean integrated squared error (MISE) criterion. Under the MISE criterion, the approximation intervals needed are three and the optimal in-sample knot positions are at [0 1 8 15]. However, the fitted curve under the MISE criterion lies above that of the ThaiBMA and both curves cannot be compared due to the use of different data sets. The curves of fitted term structure using MISE and GCV are very similar. Moreover, ThaiBMA curve lies within the narrowest band of 90% confidence interval generated from the bootstrapping of GCV and MISE. According to the GCV, the general guideline for term structure estimation is as follows. Spot fitting model should be used with cubic B-spline. The suggested in-sample knot position is [0 3 15]. With these specifications, the fitting power is high, and the estimation is fast and efficient to implement. Because of the benefits discussed, the robustness of B-spline approximation warrants it as a good alternative of the term structure of interest rates estimation.

Recommendations for further study include the comparison of B-spline fitting technique with other term structure models such as Nelson and Siegel (1985), and Svensson (1994). Also, a bond portfolio strategy can be implemented and tested with trading and filter rule by buying the underpriced bonds and short selling the overpriced bonds. Then model specification can be evaluated from the abnormal returns generated from the portfolio over a time horizon.

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